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# EQUIVALENT TRANSVERSE SHEAR STIFFNESS OF HONEYCOMB CORES

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**Abstract**—This work deals with the influence of geometry on the equivalent transverse shear stiffness of honeycomb sandwich plates. First, it presents the analytical solution for a two-dimensional basic cell of honeycomb structures by the two scale method of homogenization for periodic media. This solution gives the first order equivalent transverse shear modulus of honeycomb structures. Then the equivalent transverse shear stiffness of a regular honeycomb core is evaluated by the finite element method using a three-dimensional basic cell and the analytical solution for honeycomb structures. The equivalent transverse shear stiffness of a honeycomb core, in general, depends on the geometry of the sandwich plate. However, when the core depth is large compared to the hexagonal size, the aspect ratio of face panel thickness to hexagon wall thickness has little influence, and the transverse shear stiffness approaches the solution of honeycomb cellular structures. Based on the numerical study, an improved lower limit for the equivalent transverse shear stiffness of honeycomb cores is proposed, and an improved local shear stress of honeycombs is presented.

## 1. INTRODUCTION

Sandwich plates are efficient structures widely used in various engineering applications. One of the most common core materials for sandwich plates are honeycomb cellular structures built from isotropic metal foils. For a sandwich plate subjected to lateral loading, its honeycomb core must be stiff enough to prevent one face panel of the sandwich plate from sliding over the other. Such rigidities are called the transverse shear stiffness of the honeycomb core. When the core depth is much larger than the thickness of the face panels (most sandwich plates belong to this category), the transverse shear stiffness of the sandwich plate is contributed almost entirely by its core. For simplicity and efficiency, the cellular honeycomb core is idealized as a homogeneous material and its equivalent mechanical properties are used in analysis and design. Therefore, the knowledge of the equivalent transverse shear stiffness of honeycombs is very important for the analysis and design of sandwich plates.

In the past three decades, many researchers have studied the equivalent shear stiffness of honeycomb cores experimentally (e.g. references given by Allen, 1969; Adams and Maheri, 1993) and theoretically (Kelsey *et al.*, 1958 among others). However, the analytical solutions proposed so far do not agree well with the experimental results. All existing analytical solutions were obtained from some conventional approaches of stress analysis in conjunction with a simple averaging process. The conventional approaches are not rigorous enough to accurately homogenize honeycombs. The recently developed homogenization theory of periodic media (Bensoussan *et al.*, 1978) provides a rigorous and rational means for homogenizing periodic structures (Tong and Mei, 1992). A honeycomb sandwich plate is a structure built from a large number of periodic or nearly periodic substructures. Therefore, the homogenization theory can be used to homogenize honeycomb sandwich plates.

A honeycomb core is a periodic structure with two spatial scales: the scale of the hexagonal cell and the scale of the honeycomb itself. A sandwich plate with a honeycomb core is also a periodic structure with the same two spatial scales. However, when a honeycomb cellular structure is used as the core, it behaves differently from the pure honeycomb structure because the honeycomb core has a limited depth and the two face panels are bonded at the top and bottom of the core. Accordingly, the equivalent transverse shear

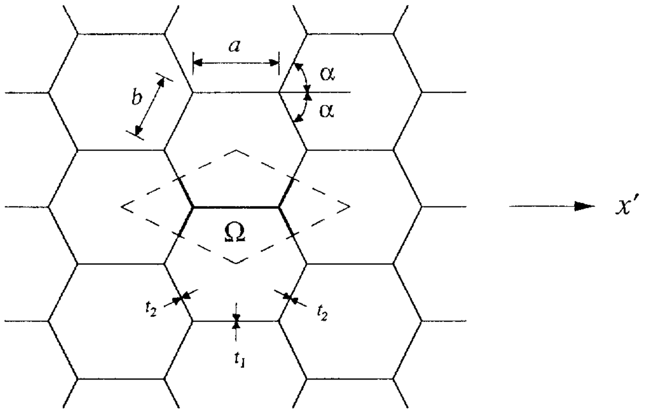


Fig. 1. The geometry of a typical honeycomb cellular structure.

stiffness of a honeycomb sandwich plate depends on the geometry of the sandwich plate. In a recent paper, Grediac (1993) attempted to numerically evaluate the influence of honeycomb depth on its equivalent transverse shear stiffness. However, he only considered stiff face panels and his approach was the unit displacement method.

The objective of this work is to study the influence of geometry on the equivalent transverse shear stiffness of honeycomb sandwich plates. To this end, the analytical solution for the equivalent transverse shear modulus of honeycomb cellular structures is presented first. A two-dimensional (2-D) basic cell is used for honeycomb structures. The analytical solution is derived by the two scale method of the homogenization theory for periodic media. Although a sandwich plate is periodic only in the plane of the plate, the "micro-strains" also vary along the normal direction of the plate. Therefore, a 3-D basic cell is needed to account for the variation of micro-strains on the normal direction. The quadrilateral strain plate element (Shi and Voyiadjis, 1991) is used to analyse the 3-D basic cell. According to the numerical results for honeycomb cores with various geometric aspect ratios, an improved lower limit for the transverse shear stiffness of sandwich plates is proposed. The present improved lower limit agrees well with the experimental results reported by other researchers (Kelsey *et al.*, 1958). Therefore, it can be used as the solution for the analysis and design of honeycomb sandwich plates.

## 2. TRANSVERSE SHEAR MODULI OF HONEYCOMB CELLULAR STRUCTURES

A honeycomb cellular structure is built from a large number of periodic or nearly periodic substructures. Such a substructure is called a basic cell in this study. Figure 1 schematically shows the geometry of a typical honeycomb structure. In the homogenization theory of periodic media, the equivalent material properties of a periodic structure can be evaluated by homogenizing its basic cell. The basic cell chosen here is also depicted in Fig. 1 where $\Omega$ signifies the area of the basic cell enclosed by the dashed line.

As an inhomogeneous material, a honeycomb structure has two spatial scales, i.e. the size of a typical hexagonal cell, named the scale $l$, and the dimension of the honeycomb, the scale $l'$. The ratio of $l/l'$ is in the order of $\varepsilon$ which is small compared to one. Usually, the thickness of a honeycomb segment is much smaller than the dimension of the basic cell, i.e. $t_1/l \ll 1$ and $t_2/l \ll 1$. If one lets $\mathbf{x}$ denote the local spatial vector of the basic cell and $\mathbf{x}'$ denote the global spatial vector of the honeycomb structure, then $\mathbf{x}'$ and $\mathbf{x}$ have the following relation:

$$\mathbf{x}' = \varepsilon \mathbf{x}. \tag{1}$$

Corresponding to the two spatial scales, the strains and stresses in a honeycomb structure can also be divided into two scales. Let $\mathbf{e}^0(\mathbf{x}')$ denote the strain tensor in the $l'$ scale and $\mathbf{e}^1(\mathbf{x}, \mathbf{x}')$ be the first order strain tensor in the $l$ scale in the basic cell; then the

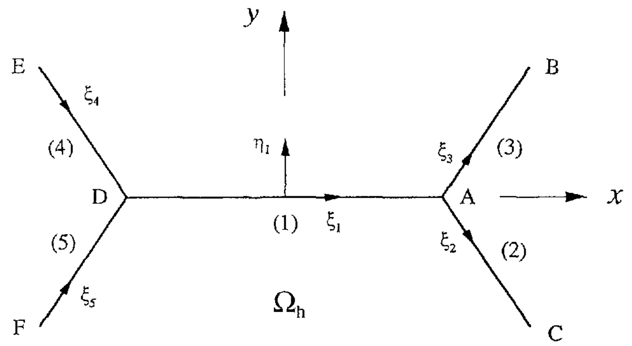


Fig. 2. 2-D basic cell of honeycomb structures.

perturbation method leads to the equilibrium equations in the basic cell (Tong and Mei, 1992; Shi and Tong, 1994a)

$$\nabla \cdot \boldsymbol{\sigma} = \nabla \cdot \mathbf{A} : (\mathbf{e}^0 + \mathbf{e}^1) = \mathbf{0} \quad \text{in } \Omega_h \tag{2}$$

$$\mathbf{A} : (\mathbf{e}^0 + \mathbf{e}^1) \cdot \mathbf{n} = \mathbf{0} \quad \text{on } \partial\Omega_h \tag{3}$$

together with the periodic conditions of the basic cell, which are the important constraints resulting from the homogenization theory of periodic media. In the equations above, $\mathbf{A}$ is the elasticity coefficient tensor; $\Omega_h$ denotes the domain occupied by the honeycomb segments of the basic cell in the $x$–$y$ plane; $\partial\Omega_h$ represents the boundaries of $\Omega_h$; $\mathbf{n}$ is the direction cosine vector of $\partial\Omega_h$; the symbol : signifies the tensor contraction operator; and the stress–strain relation

$$\boldsymbol{\sigma}(\mathbf{x}, \mathbf{x}') = \mathbf{A} : [\mathbf{e}^0(\mathbf{x}') + \mathbf{e}^1(\mathbf{x}, \mathbf{x}')] \tag{4}$$

is utilized. The $l$ scale strains $\mathbf{e}^1$ appearing in eqns (2) and (3), which can be considered as the "micro-strains" of the basic cell, can be solved as a boundary value problem subjected to initial strains $\mathbf{e}^0$ and the periodic conditions of the homogenization equations.

A honeycomb structure is periodic in the $x$–$y$ plane only. The 2-D model shown in Fig. 2 implies that all variables are independent of the $z$-axis. In order to derive an analytical solution, only the honeycombs made of isotropic materials are considered in the present work.

For the basic cell of a honeycomb cellular structure, it is convenient to evaluate the strains in the local coordinates of the segments. Because of the assumption of isotropic materials, only transverse shear strains in each segment need to be evaluated. Corresponding to uniform global transverse shear strains $e^0_{zx}$ and $e^0_{zy}$, the initial shear strains of segment $i$ in its local coordinates take the form

$$e^0_{z\xi i} = e^0_{zx} \cos\alpha_i + e^0_{zy} \sin\alpha_i \tag{5}$$

$$e^0_{z\eta i} = -e^0_{zx} \sin\alpha_i + e^0_{zy} \cos\alpha_i \tag{6}$$

where $\xi_i$ and $\eta_i$ are, respectively, the tangential and normal coordinates of segment $i$ as illustrated in Fig. 2; $\alpha_i$ signifies the angle measured from the $x$-axis of the basic cell to $\xi_i$. Substitution of eqn (6) into the traction free condition given in eqn (3) leads to

$$e^1_{z\eta i} = -e^0_{z\eta i} = e^0_{zx} \sin\alpha_i - e^0_{zy} \cos\alpha_i. \tag{7}$$

The displacement resulting from a constant $e^1_{z\eta i}$ is independent of $\xi_i$. Consequently, the local strain component given in eqn (7) satisfies the displacement periodicity.

The linear strain–displacement relation is of the form

$$2e^{1}_{z\xi i} = \frac{\partial u_i^{(1)}}{\partial z} + \frac{\partial w_i^{(1)}}{\partial \xi_i} = \frac{\partial w_i^{(1)}}{\partial \xi_i} \tag{8}$$

in which $w_i^{(1)}(\xi)$ is the warping of segment $i$ in the $l$ scale along the segment axis. Since $w_i^{(1)}(\xi)$ affects the displacement periodicity in $\Omega$, $e^{1}_{z\xi i}$ will be computed from the assumed displacement $w_i^{(1)}(\xi)$.

For a given constant initial strain field, eqn (2) indicates that a constant local strain field can satisfy the equilibrium equation. Therefore, it is feasible to assume the warping of segment $i$ in its local coordinates $w_i^{(1)}$ in the form

$$w_i^{(1)} = s_i\xi_i + c_i \quad (i = 1,2,3,4,5;\text{no summation in } i). \tag{9}$$

To prevent the rigid body motion, the center of the basic cell is assumed to be supported, that is

$$w_1^{(1)}(0) = c_1 = 0. \tag{10}$$

By using the periodic conditions of the basic cell

$$w_2^{(1)}|_{\mathrm{C}} = w_4^{(1)}|_{\mathrm{E}}, w_3^{(1)}|_{\mathrm{B}} = w_5^{(1)}|_{\mathrm{F}} \tag{11}$$

$$e^{1}_{z\xi_2}|_{\mathrm{C}} = e^{1}_{z\xi_4}|_{\mathrm{E}}, e^{1}_{z\xi_3}|_{\mathrm{B}} = e^{1}_{z\xi_5}|_{\mathrm{F}} \tag{12}$$

and the displacement continuity as well as the equilibrium conditions at points A and B, one can obtain (Shi and Tong, 1994a)

$$s_1 = \frac{1}{t_1+2t_2a/b}(2t_2\operatorname{con}\alpha - t_1)2e^{0}_{zx} \tag{13}$$

$$s_2 = s_3 = s_4 = s_5 = -\frac{a}{b}s_1 \tag{14}$$

$$c_2 = c_3 = \frac{a}{2}s_1, c_4 = c_5 = 0. \tag{15}$$

Equation (13) indicates that the local warping is independent of $e^{0}_{zy}$. This is because the geometry and the material of the basic cell under consideration are symmetric about the $x$-axis. Equations (9) and (13)–(15) show that the local warping of the basic cell is symmetric about the $x$-axis and anti-symmetric about the $y$-axis. It should be pointed out that the warping of the basic cell as a whole given in this work is rigorously derived from the homogenization theory and valid for honeycombs with general configuration, while the one given by Grediac (1993), called rotation there, is only valid for the honeycomb with equal length, i.e. $a = b$.

It follows from eqns (8) and (9) that

$$2e^{1}_{z\xi i} = s_i. \tag{16}$$

Using the strain transformation, one can write the strains of segment $i$ in the basic cell's coordinates as

$$(e^1_{zx})_i = e^1_{z\xi i}\cos\alpha_{\xi i} + e^1_{z\eta i}\sin\alpha_{\xi i} \tag{17}$$

$$(e^1_{zy})_i = -e^1_{z\xi i}\sin\alpha_{\xi i} + e^1_{z\eta i}\cos\alpha_{\xi i} \tag{18}$$

where $\alpha_{\xi i}$ denotes the angle from the $\xi$-axis of segment $i$ to the $x$-axis of the basic cell. The theory of homogenization for periodic media gives the average transverse shear stress over the basic cell $\langle\sigma_{zx}\rangle$ as

$$\begin{aligned}\langle\sigma_{zx}\rangle &= \frac{1}{\Omega}\int_{\Omega_h}\sigma_{zx}\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{\Omega}\int_{\Omega_h} G\,2[e^0_{zx}+e^1_{zx}(x,y)]\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{G}{\Omega}\sum_{i=1}^{5} t_i l_i 2[e^0_{zx}+e^1_{z\xi i}\cos\alpha_{\xi i}+e^1_{z\eta i}\sin\alpha_{\xi i}] \\ &= \frac{G}{\Omega}[at_1+2bt_2(1-\sin^2\alpha)+\frac{2t_2\cos\alpha - t_1}{t_1+2t_2a/b}(at_1-2bt_2\cos\alpha)]2e^0_{zx}\end{aligned} \tag{19}$$

in which $G$ is the shear modulus of the isotropic honeycomb; $t_i$ and $l_i$ are, respectively, the thickness and the length of segment $i$; and eqns (4), (7), (13) and (16) are used. If one lets $G_{zx}$ be the equivalent transverse shear modulus in the $x$-direction of the basic cell, then the average stress can be expressed in terms of the strain in the $l'$ scale as

$$\langle\sigma_{zx}\rangle = G_{zx}2e^0_{zx}. \tag{20}$$

Consequently, it follows from eqns (19) and (20) that the equivalent transverse shear modulus of honeycombs is of the form

$$G_{zx} = \frac{G}{\Omega}[at_1+2bt_2(1-\sin^2\alpha)+\frac{2t_2\cos\alpha-t_1}{t_1+2t_2a/b}(at_1-2bt_2\cos\alpha)]. \tag{21}$$

Similarly, one can obtain

$$G_{zy} = \frac{G}{\Omega}2bt_2(1-\cos^2\alpha). \tag{22}$$

In eqn (21), the last term is the portion resulting from the warping of the basic cell $w_i^{(1)}(\xi)$, while $G_{zy}$ is independent of this warping since both the basic cell and its warping are symmetric about the $x$-axis.

The regular honeycomb in which $a = b$, $t_1 = 2t_2 = 2t$ and $\alpha = 60°$ is the most common shape for honeycomb structures in various engineering applications. Consequently, its equivalent transverse shear modulus has been studied by many researchers. Substituting the geometric parameters into eqns (21) and (22), one obtains

$$G_{zx} = \left(\frac{5}{3}-\frac{1}{6}\right)\frac{t}{d}G = \frac{3}{2}\frac{t}{d}G \tag{23}$$

$$G_{zy} = \frac{t}{d}G \tag{24}$$

where $\Omega = \frac{3}{2}ad$ is used and $d = \sqrt{3}a$ is the diameter of the inscribed circle of a regular hexagon. In eqn (23), $[-\frac{1}{6}(t/d)G]$ is the contribution of the basic cell's warping. In the last three decades, the paper written by Kelsey *et al.* (1958) is the principal work on the equivalent transverse shear stiffness of honeycombs. It is based on the conventional approaches of structural analysis. Kelsey *et al.* (1958) gave the lower limit by the unit force

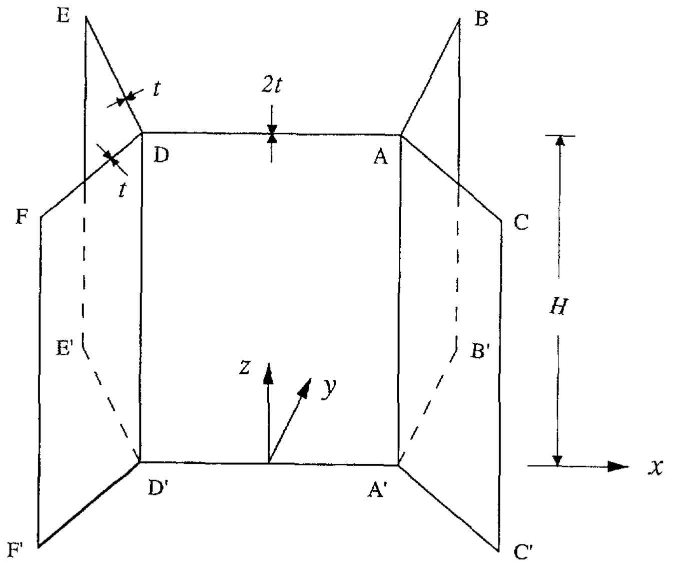


Fig. 3. 3-D basic cell of honeycomb cores.

method and obtained the upper limit from the unit displacement method. For regular honeycombs with equal walls, the lower and the upper limits of $G_{zy}$ are identical and yield the same value as eqn (24). The lower limit of $G_{zx}$ given by Kelsey *et al.* (1958) is the same as that in eqn (23). The upper limit of $G_{zx}$ is

$$(G_{zx})_{\text{upper}} = \frac{5}{3}\frac{t}{d}G. \tag{25}$$

It can be seen from eqn (23) that this upper limit corresponds to the result obtained in the present study where the warping of the basic cell as a whole is neglected. Therefore, the present study can illustrate the lower and upper limits given by Kelsey *et al.* (1958).

## 3. THE INFLUENCE OF HONEYCOMB DEPTH ON ITS TRANSVERSE SHEAR MODULUS

The 2-D model considered above implies that the depth of the honeycomb is much larger than the dimension of the basic cell, i.e. $l/H \ll 1$, where $H$ signifies the honeycomb depth. When $H$ is in the same order as $l$, it is obvious that the linear local warping along a segment used in the 2-D model might not be feasible and the local warping could also vary in the normal direction of the basic cell. Consequently, the contribution of the local warping to $G_{zx}$ in this case will be different from the last term in eqn (21) given by the 2-D model. In order to take into account the variation of $e^1_{zx}$ in the $z$-direction, a 3-D model must be employed. The 3-D basic cell used in this study is illustrated in Fig. 3. Because of the complexity in the computation of 3-D models, the 3-D basic cell will be studied by the finite element method.

Since $G_{zy}$ is independent of local warping, $G_{zy}$ given by a 3-D model will be the same as that obtained from the 2-D basic cell. Therefore, only $G_{zx}$ needs to be evaluated from the 3-D model. The regular hexagonal cell with $t_1 = 2t_2 = 2t$ is the most widely used shape for honeycomb cores. The present numerical study of the honeycomb depth's influence on its shear modulus is confined to the regular honeycombs. For the 3-D basic cell shown in Fig. 3, the average transverse shear stress is defined as

$$\langle\sigma_{zx}\rangle = \frac{G}{\Omega\cdot H}\int_{\Omega_h}\int_0^H 2[e^0_{zx}+e^1_{zx}(x,y,z)]\,dx\,dy\,dz = G_{zx}\,2e^0_{zx} \tag{26}$$

where the transverse shear strain $e^1_{zx}$ in each plate of the basic cell can be calculated from $e^1_{z\xi i}$ which takes the form

Table 1. The influence of honeycomb depth on deflection and shear modulus

| $H/a$ | 1 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| $\bar{w}/\bar{w}_0$ | 1.537 | 1.307 | 1.227 | 1.177 | 1.145 | 1.125 |
| $\beta$ | −0.112 | −0.111 | −0.111 | −0.111 | −0.110 | −0.110 |

$$2e^1_{z\xi i} = \frac{\partial u_i^{(1)}}{\partial z} + \frac{\partial w_i^{(1)}}{\partial \xi_i}. \tag{27}$$

It should be noticed that the first term in the right hand side of the above equation is not equal to zero in a 3-D model.

The contribution of the local warping given by the 2-D model to $G_{zx}$ is the second term in eqn (23). Let $\beta$ denote the non-dimensional contribution of the local warping in a 3-D basic cell, then the general form of $G_{zx}$ for regular honeycombs can be written as

$$G_{zx} = \left(\frac{5}{3} + \beta\right)\frac{t}{d}G \tag{28}$$

where $\beta$ is of the form

$$\beta = \frac{2}{3t \cdot a \cdot H}\sum_{i=1}^{5} t_i \int_0^H \int_{l_i} (e^1_{z\xi i}/e^0_{zx}) \cos\alpha_{\xi i}\, \mathrm{d}\xi_i\, \mathrm{d}z. \tag{29}$$

The modifying parameter $\beta$ is a function of the ratio $H/a$. The task in this section is to investigate the relation between $\beta$ and $H/a$.

The honeycomb structure considered in this study is built from thin foils with $a/t = 72$. The basic cell shown in Fig. 3 is modeled by the four-noded strain plate element developed by Shi and Voyiadjis (1991). A $6 \times 8$ mesh is used for plate DAA′D′ and a $3 \times 8$ mesh for each of other plates when $H/a \leqslant 6$, and $4 \times 8$ for DAA′D′ $3 \times 8$ for each of other plates when $H/a \geqslant 8$. This mesh is fine enough since the mesh of $4 \times 6$ for DAA′D′ and $2 \times 6$ for others yields the same result. Because the displacement is nonsymmetric about the $y$-axis, as indicated by eqns (9) and (14), it cannot only take one quarter of the basic cell to compute its stress state even though the geometry of the basic cell is symmetric about both the $x$- and the $y$-axes. In order to enforce the periodic conditions, the whole basic cell is considered in the present numerical analysis.

For given global strains $e^0_{zx}$ and $e^0_{zy}$, the corresponding initial strains of each plate in its local coordinates can be calculated from eqns (5) and (6). Accordingly, the equivalent loads acting on the basic cell can also be expressed in terms of these global strains and the basic cell's geometry. The periodic conditions require that displacements and stresses along BB′ equal to those along FF′ and all quantities on CC′ equal to those on EE′.

The numerical analysis of the 3-D model shows that $\partial u_i^{(1)}/\partial z$ is not equal to zero as given by the 2-D model, and it has a considerable contribution to $\beta$. Table 1 tabulates the finite element solutions of $\beta$ versus $H/a$. The ratios of $\bar{w}/\bar{w}_0$ are also given in the table, where $\bar{w}$ and $\bar{w}_0$ are the maximum deflections of the local warping obtained from the 3-D and 2-D models, respectively. The maximum deflections occur at the two joints of the basic cell. Equations (9) and (13) give

$$\bar{w}_0 = \frac{1-\cos\alpha}{1+a/b}\frac{a}{2}(2e^0_{zx}) \tag{30}$$

in which $t_1 = 2t_2$ is used. This table indicates that when $H$ is close to $a$, the real warping of the honeycomb is much larger than that predicted by the 2-D basic cell, but as $H/a$ increases, $\bar{w}$ decreases quite fast and approaches a value close to $\bar{w}_0$. However, $\beta$ given by the 3-D model is quite stable and different from the value obtained from the 2-D model. This is because in the 2-D model the variation of $e^1_{zx}$ on the $z$-axis is neglected and a linear function is assumed for $w^{(1)}$. Nevertheless, the 2-D model not only enables one to obtain an analytical solution, but also yields a good approximation of both $w^{(1)}$ and $G_{zx}$. When $H/a$ is large enough, it follows from eqn (28) that the 3-D model gives the equivalent transverse shear modulus of regular honeycomb structures as

$$G_{zx} = \left(\frac{5}{3} - 0.11\right)\frac{t}{d}G = 1.56\frac{t}{d}G. \tag{31}$$

## 4. INFLUENCE OF GEOMETRY OF SANDWICH PLATES ON $G_{zx}$

In eqn (28), $\beta$ results from the local warping of honeycomb cellular structures. When a honeycomb is used as the core material of a sandwich plate, the honeycomb cannot warp freely as in the case of honeycomb cellular structures, since the warping is constrained by the bending rigidity of the sandwich's face panels. Consequently, the transverse shear stiffness of honeycomb sandwich plates could be different from that of a honeycomb cellular structure. This is also true when the transverse shear stiffness of the face panels is negligible.

A sandwich plate with a honeycomb core is also a periodic structure with two spatial scales. However, in this case the basic cell must include the face panels to account for the effect of sandwich face panels on $G_{zx}$. The computational model used in this study is the 3-D hexagon cell depicted in Fig. 3 together with two diamond plates bonded on the top and bottom of the cell. It is assumed that the bonding between the honeycomb core and the face panels is perfect. If the face panels are to be treated as 2-D plates rather than a 3-D solid, the thickness of the face panels $h$ must be much smaller than the depth of the sandwich core, i.e. $h/H \ll 1$. Fortunately, this is the case for the sandwich plates used in various engineering applications.

The local bending rigidity of a sandwich face panel made of isotropic material $D_f$ is of the form

$$D_f = \frac{E_f h^3}{12(1-\nu_f^2)} \tag{32}$$

in which $E_f$ and $\nu_f$ are, respectively, the Young's modulus and Poisson's ratio of the sandwich face panels. The stiffness ratio of honeycomb core to face panel is defined by a non-dimensional parameter

$$\gamma = \frac{E \cdot t \cdot a \cdot H}{12 D_f} \tag{33}$$

where $E$ is the Young's modulus of the honeycomb core. A strong face panel leads to a small $\gamma$ and vice versa.

The 3-D basic cell for honeycomb sandwich plates is also studied by the finite element method. The mesh layout for the face panels is depicted in Fig. 4. The mesh used in the analysis of honeycomb structures is employed again for the honeycomb core portion of the basic cell. The finite element for the face panels is also the quadrilateral strain element (Shi and Voyiadjis, 1991) which can automatically reduce to the corresponding triangular element. The periodic conditions should be applied to both the honeycomb core and the face panels. The equivalent loads in this case can also be written in terms of $e^0_{zx}$ and $e^0_{zy}$.

For a fixed $h/t = 10$, the finite element solutions for $\bar{w}_f/\bar{w}_0$ and $\beta$ versus $H/a$ are listed in Table 2. It should be noticed that in this case the deflection $w$ across the core depth is not constant; $\bar{w}_f$ is the maximum deflection of the face panel. Similar to the honeycomb structures presented in the previous section, $\partial u_i^{(1)}/\partial z$ also has a considerable contribution to $\beta$. Because of the assumption of thin face panels, the transverse shear strains in the face

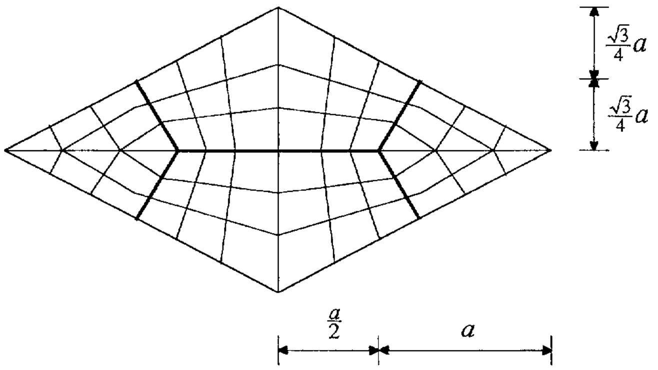


Fig. 4. Mesh layout for the face panels in the basic cell of sandwich plates.

Table 2. The effect of honeycomb depth on $G_{zx}$ of sandwich plates ($h/t$ = 10)

| $H/a$ | 1 | 2 | 3 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|---|
| $\gamma$ | 2.41 | 4.82 | 7.23 | 9.64 | 14.47 | 19.27 | 24.10 |
| $\bar{w}_f/\bar{w}_0$ | 0.298 | 0.342 | 0.347 | 0.350 | 0.355 | 0.358 | 0.360 |
| $\beta$ | −0.0555 | −0.0793 | −0.0896 | −0.0948 | −0.100 | −0.103 | −0.104 |

Table 3. The influence of bending rigidity of sandwich faces on $G_{zx}$ ($H/a = 6$)

| $h/t$ | 0 | 2.5 | 5 | 10 | 15 | 20 | 40 |
|---|---|---|---|---|---|---|---|
| $\gamma$ | $\infty$ | 926.08 | 115.76 | 14.47 | 4.29 | 1.81 | 0.226 |
| $\bar{w}_f/\bar{w}_0$ | 1.177 | 1.057 | 0.836 | 0.355 | 0.140 | 0.0643 | 0.0085 |
| $\bar{w}_h/\bar{w}_0$ | 1.177 | 1.140 | 1.179 | 1.224 | 1.239 | 1.243 | 1.246 |
| $\beta$ | −0.111 | −0.111 | −0.108 | −0.100 | −0.0965 | −0.0953 | −0.0945 |

panels are neglected. Table 2 indicates that the warping of the outer surfaces of a honeycomb sandwich plate is indeed much smaller than that of the honeycomb structure, and $\beta$ for sandwich plates, in general, depends on the aspect ratio of $H/a$. However, when $H/a > 6$, both $\bar{w}_f/\bar{w}_0$ and $\beta$ remain basically unchanged and converge to some constants. Therefore, $H/a = 6$ can be taken as the limit case. $\bar{w}_f/\bar{w}_0$, $\bar{w}_h/\bar{w}_0$ and $\beta$ versus $h/t$ for a fixed $H/a = 6$ are given in Table 3 in which $H/h \geqslant 10$ in all cases. As mentioned earlier, the deflection is varying across the core depth. The contour of the deflection across D′D (see Fig. 3), for the case of $h/t = 40$, is depicted in Fig. 5, which is obtained by a constant in-plane strain element. In this table $\bar{w}_h$ is the maximum deflection inside of the honeycomb core, and $h/t = 0$ represents the cellular structure of honeycomb. The location of $\bar{w}_h$ is mesh dependent. However, this is insignificant since the face panels only have a local effect on the

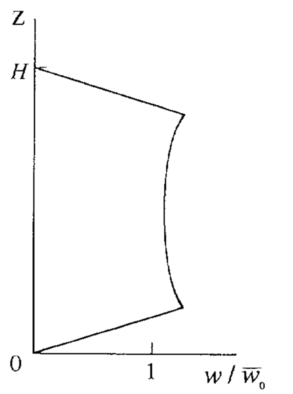


Fig. 5. The deflection across the core depth D′D.

honeycomb's warping as will be shown below. Figure 5 and Table 3 clearly illustrate that the warping in the interior of the sandwich core does not change much from that of honeycomb cellular structures and the modifying parameter of shear modulus $\beta$ hardly changes from the value of the honeycomb either, even though the core is constrained by face panels and the warping of the sandwich plate is vanishing as the stiffness ratio $\gamma$ decreases. This indicates that the influence of face panels on the honeycomb's warping is quite localized.

Tables 2 and 3 show that all $\beta$ are larger than or equal to $-0.111$ which is the value of honeycomb cellular structures given by the 3-D basic cell. Therefore, the equivalent transverse shear modulus of honeycomb structures given in eqn (31) is an improved lower limit for $G_{zx}$ of sandwich plates with regular honeycomb cores. That is

$$(G_{zx})_{\text{lower}} = 1.56\frac{t}{d}G. \tag{34}$$

When the ratio of the honeycomb depth to the hexagon length of honeycomb core, $H/a$, is larger than 6, this lower limit is very close to the real value. The experimental results given by Kelsey *et al.* (1958) show that the longitudinal equivalent transverse shear stiffness of honeycomb sandwich plates really falls in the range bounded by $\frac{3}{2}(t/d)G$ and $\frac{5}{3}(t/d)G$. On the other hand, the present solution of static moduli is larger than the experimental dynamic moduli given by Adams and Maheri (1993).

By setting the deflection of the top line of the basic cell equal to zero, Grediac (1993) implies that the face panels are extremely stiff, which corresponds to $\gamma = 0$ in the present notation. For the regular honeycomb considered here, Grediac's result converges to the lower limit given by Kelsey *et al.* (1958), which is identical to eqn (23), as $H/a$ increases. However, Table 3 indicates that even in this special case the present solution is different from that obtained by Grediac. This can be attributed to the fact that Grediac only used one quarter of the basic cell for the nonsymmetric warping of the honeycomb core. The nonsymmetric displacement in the $z$-direction given by the present work is depicted in Fig. 5.

Even though the 3-D numerical solution for the equivalent transverse shear stiffness given in eqn (34) does not change much from the 2-D analytical solution in eqn (23), the local stress obtained from a 3-D model is significantly different from that of a 2-D model. It follows from eqns (4), (5) and (27) that the local shear stress in panel $i$ takes the form

$$\sigma_{z\xi i} = G\left[2(e^0_{zx}\cos\alpha_i + e^0_{zy}\sin\alpha_i) + \frac{\partial u_i^{(1)}}{\partial z} + \frac{\partial w_i^{(1)}}{\partial \xi_i}\right]. \tag{35}$$

In the 2-D model $(\partial u_i^{(1)}/\partial z) = 0$, while in the 3-D model $(\partial u_i^{(1)}/\partial z)$, which is dependent on the geometry of sandwich plates, is not equal to zero but in the same order as $(\partial w_i^{(1)}/\partial \xi_i)$. This local stress given by the 3-D model is very important for the local behavior of honeycombs, such as the local buckling of honeycombs under transverse shear forces (Shi and Tong, 1994b)

## 5. SUMMARY AND CONCLUSIONS

In this work, the sandwich plate with honeycomb cores is treated as a periodic structure with two spatial scales. By using the two scale method of homogenization for periodic media, this paper evaluates the equivalent transverse shear stiffness of regular honeycomb cores. A 2-D basic cell is chosen to derive the analytical solution of honeycomb cellular structures, which provides a reference for the study of honeycomb sandwich plates. A 3-D basic cell comprised of a 3-D hexagon cell and face panels is used to model transverse shear behavior of sandwich plates. The 3-D basic cell is studied by the finite element method. The various numerical results presented here indicate the following.

(1) Although a honeycomb cellular structure is periodic only in one plane, the strains in the $l$ scale also vary in the normal direction of the plane. Therefore, a 3-D model is needed to accurately evaluate the equivalent transverse properties of honeycombs.
(2) For honeycomb cellular structures, the warping in the $l$ scale is a function of $H/a$, the ratio of honeycomb depth to hexagon dimension. Nevertheless, the equivalent transverse shear moduli are almost independent of $H/a$.
(3) The face panels on a honeycomb core indeed constrain the warping of the outer surfaces of the core. But, the influence of face panels on the honeycomb's warping is quite localized.
(4) In general, the longitudinal shear stiffness $G_{zx}$ of honeycomb cores depends on the geometry of the hexagon and face panels of sandwich plates. However, when $H/a \geqslant 6$, $G_{zx}$ is quite close to a constant which equals the corresponding modulus of honeycomb cellular structures. Consequently, the equivalent transverse shear modulus of honeycombs given in eqn (31) can be chosen as the improved lower limit for the equivalent transverse shear stiffness $G_{zx}$ of honeycomb cores. The accuracy of this lower limit is quite good for the honeycomb sandwich plates where $H/a \geqslant 6$.

Only the equivalent transverse shear stiffness of honeycomb cores is considered in the present study. By employing the homogenization theory, the authors presented the analytical solution for the equivalent in-plane moduli of honeycomb structures (Shi and Tong, 1994a).

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